IV International Conference on Particle-based Methods – Fundamentals and Applications
PARTICLES 2015
E. Oñate, M. Bischoff, D.R.J. Owen, P. Wriggers & T. Zohdi (Eds)

THE ALGORITHM FOR 3D VORTEX SHEET INFLUENCE COMPUTATION IN MESHLESS VORTEX METHOD

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Key words: Vortex Element Method, Vorticity Flux, Integral Equation, Quadrature Formula, Singularity Exclusion

Abstract. Vortex element method (VEM) is very suitable when solving FSI-problems, especially two-way coupled hydroelastic problems, when it is important to simulate unsteady body motion in incompressible flow [1]. Pure Lagrangian modifications of vortex methods don't require mesh generation in the fluid domain and its reconstruction at every time step due to body's motion. Vortex methods need much less computational resources (time of computations, memory and disk space) in comparison with most popular mesh methods.

Flow simulation using VEM requires the solution of two main problems: vortex wake evolution simulation and vorticity flux on the body surface computation, which are solved sequentially, step by step at every time step. In order to simulate vortex wake evolution there are developed number of approaches: vortex wake approximation with vortex filaments, vortex lattices, vortex points (vortons) of different types etc. The vortex fragmenton model is developed by authors and it has been applied to some FSI-problems [2]. As to vorticity flux simulation, one of the most important problems is vortex sheet intensity computation. There are two fundamental approaches, which are based on elimination of the limit values of normal or tangential velocity components on the body surface [3].

The accuracy of 'normal' approach in some FSI-applications is not enough for practice. In 2D-case 'tangent' approach allows to obtain much more accurate results, but it requires more precise discretization and integration schemes usage [3]. Such schemes are constructed and investigated by authors for 2D-case [4].

In the present research 3D-case is considered. The discretization scheme for 'tangent' approach is presented. The body surface is triangulated using arbitrary mesh generator and vortex sheet intensity assumed to be piecewise constant on the cells. 'Tangent' approach includes calculation of two surface integrals: the integral over the 'influence' cell and the integral over the 'control' cell. For arbitrary triangular mesh 'influence' integral can be calculated analytically. Numerical calculation of the 'control' integral using Gaussian points leads to significant errors for neighbor cells. The quadrature formula with

analytical singularity integration is derived. Non-singular part is integrated numerically by using Gaussian quadratures. All necessary formulae are obtained in invariant form and can be applied for arbitrary triangular cells.

The original algorithm which is based on the developed approach for vortex influences computation is developed.

1 INTRODUCTION AND PROBLEM STATEMENT

The problem of 3D incompressible flow simulation around bluff body is considered. The governing equations are Navier — Stokes equations

$$\nabla \cdot \boldsymbol{V} = 0, \quad \frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = \nu \nabla^2 \boldsymbol{V} - \frac{\nabla p}{\rho_\infty},$$

with boundary conditions

$$\lim_{r\to\infty} \boldsymbol{V} = \boldsymbol{V}_\infty, \quad \lim_{r\to\infty} p = p_\infty, \quad \boldsymbol{V}(\boldsymbol{r},\, t)\Big|_{\boldsymbol{r}\in K} = 0,$$

where $\boldsymbol{V}$ is flow velocity; p — pressure; $\rho_\infty = \text{const}$ — density; ν — kinematic viscosity coefficient; $\boldsymbol{V}_\infty$ and p_∞ are parameters of the incident flow; K is body's surface.

The viscosity assumed to be small, so according to L. Prandtl's theory it is possible to take its influence into account only as a cause of vorticity generation on body surface. The flow can be considered inviscid, with vorticity flux from the surface.

The immovable streamlined body is simulated by vortex sheet $\boldsymbol{\gamma}(\boldsymbol{r},\, t)$ influence, which is placed on the body surface, $\boldsymbol{r} \in K$. The vorticity flux can be simulated if this vortex sheet is free; that means that at every time step this sheet is split into separate vortex elements which form vortex wake around the body.

Vortex wake evolution can be simulated by using one of Lagrangian vortex element methods [1, 2].

As to vorticity flux simulation, one of the most important problems is vortex sheet intensity computation. There are two fundamental approaches, which are based on elimination of the limit values of normal or tangential velocity components on the body surface [3]. These approaches are denoted as 'normal' and 'tangent'.

The accuracy of 'normal' approach, especially in FSI-applications, when streamlined bodies are movable and deformable, sometimes is not enough for practice. In 2D-case 'tangent' approach allows to obtain much more accurate results, but it requires more precise discretization and integration schemes usage [3]. Such schemes are constructed and investigated by authors for 2D-case [4].

The aim of the present research is development of the corresponding numerical schemes for 3D case.

2 INTEGRAL EQUATION FOR VORTEX SHEET INTENSITY

Due to vortex sheet on the body surface, velocity field has the discontinuity, and its limit value from body side is

$$\boldsymbol{V}_-(\boldsymbol{r},\,t) = \boldsymbol{V}_\infty + \frac{1}{4\pi}\int\limits_{S(t)} \frac{\boldsymbol{\Omega}(\boldsymbol{\xi},\,t)\times(\boldsymbol{r}-\boldsymbol{\xi}\,)}{\left|\boldsymbol{r}-\boldsymbol{\xi}\,\right|^3}d\xi + \\ + \frac{1}{4\pi}\int\limits_{K} \frac{\boldsymbol{\gamma}(\boldsymbol{\xi},\,t)\times(\boldsymbol{r}-\boldsymbol{\xi}\,)}{\left|\boldsymbol{r}-\boldsymbol{\xi}\,\right|^3}d\xi - \frac{\boldsymbol{\gamma}(\boldsymbol{r},\,t)\times\boldsymbol{n}(\boldsymbol{r}\,)}{2}, \quad \boldsymbol{r}\in K.$$

Here $S(t)$ is vortex wake region; $\boldsymbol{n}(\boldsymbol{r})$ is unit outer normal vector on body surface K.

In order to satisfy the boundary condition on the body surface, vortex sheet intensity should satisfy the integral equation $\boldsymbol{V}_-(\boldsymbol{r},\,t)=0$, $\boldsymbol{r}\in K$.

As it proved in [3], it is enough to satisfy this equation only for tangent component of the velocity:

$$\boldsymbol{n}(\boldsymbol{r}\,)\times\Big(\boldsymbol{V}_-(\boldsymbol{r},\,t)\times\boldsymbol{n}(\boldsymbol{r}\,)\Big)=0.$$

It leads to Fredholm-type integral equation of the 2-nd kind

$$\frac{\boldsymbol{n}(\boldsymbol{r}\,)}{4\pi}\times\left(\int\limits_K \frac{\boldsymbol{\gamma}(\boldsymbol{\xi},\,t)\times(\boldsymbol{r}-\boldsymbol{\xi}\,)}{\left|\boldsymbol{r}-\boldsymbol{\xi}\,\right|^3}\times\boldsymbol{n}(\boldsymbol{r}\,)d\xi\right) - \frac{\boldsymbol{\gamma}(\boldsymbol{r},\,t)\times\boldsymbol{n}(\boldsymbol{r}\,)}{2} = \boldsymbol{f}(\boldsymbol{r},\,t), \quad \boldsymbol{r}\in K, \quad (1)$$

where

$$\boldsymbol{f}(\boldsymbol{r},\,t) = -\boldsymbol{n}(\boldsymbol{r})\times\left(\boldsymbol{V}_\infty + \frac{1}{4\pi}\int\limits_{S(t)}\frac{\boldsymbol{\Omega}(\boldsymbol{\xi},\,t)\times(\boldsymbol{r}-\boldsymbol{\xi}\,)}{\left|\boldsymbol{r}-\boldsymbol{\xi}\,\right|^3}\,d\xi\right)\times\boldsymbol{n}(\boldsymbol{r}\,)$$

is known vector function.

3 INTEGRAL EQUATION DISCRETIZATION

In order to find approximate solution of hypersingular integral equation (1), the following assumptions can be made:

1. Body surface is triangulated into N flat cells K_i with areas A_i and normal vectors $\boldsymbol{n}_i$, $i=1,\ldots,N$.

2. The unknown vortex sheet intensity on the i-th cell is constant vector $\boldsymbol{\gamma}_i$, $i=1,\ldots,N$, which lies in the plane of the i-th cell, i.e. $\boldsymbol{\gamma}_i\cdot\boldsymbol{n}_i=0$.

3. The integral equation (1) is satisfied on average on the cells.

According to these assumptions the discrete analogue of equation (1) can be derived:

$$\frac{1}{4\pi A_i}\sum_{j=1}^{N}\int\limits_{K_i}\left(\int\limits_{K_j}\boldsymbol{n}_i\times\left(\frac{\boldsymbol{\gamma}_j\times(\boldsymbol{r}-\boldsymbol{\xi}\,)}{\left|\boldsymbol{r}-\boldsymbol{\xi}\,\right|^3}\times\boldsymbol{n}_i\right)d\xi\right)dr-\frac{\boldsymbol{\gamma}_i\times\boldsymbol{n}_i}{2}=\frac{1}{A_i}\int\limits_{K_i}\boldsymbol{f}(\boldsymbol{r},\,t)dr,\quad i=1,\ldots,N. \tag{2}$$

To write down (2) in form of linear algebraic system we should choose local orthonormal basis on every cell $(\boldsymbol{\tau}_i^{(1)},\,\boldsymbol{\tau}_i^{(2)},\,\boldsymbol{n}_i)$, where tangent vectors $\boldsymbol{\tau}_i^{(1)},\,\boldsymbol{\tau}_i^{(2)}$ can be chosen arbitrary and $\boldsymbol{\tau}_i^{(1)}\times\boldsymbol{\tau}_i^{(2)}=\boldsymbol{n}_i$. So $\boldsymbol{\gamma}_i=\gamma_i^{(1)}\boldsymbol{\tau}_i^{(1)}+\gamma_i^{(2)}\boldsymbol{\tau}_i^{(2)}$ and we can project (2) for every i-th panel on directions $\boldsymbol{\tau}_i^{(1)}$ and $\boldsymbol{\tau}_i^{(2)}$:

$$\begin{aligned}
&\frac{1}{4\pi A_i}\boldsymbol{\tau}_i^{(1)}\cdot\left(\sum_{j=1}^{N}\gamma_j^{(1)}\boldsymbol{\nu}_{ij}^{(1)}+\sum_{j=1}^{N}\gamma_j^{(2)}\boldsymbol{\nu}_{ij}^{(2)}\right)-\frac{\gamma_i^{(2)}}{2}=\frac{b_i^{(1)}}{A_i},\\
&\frac{1}{4\pi A_i}\boldsymbol{\tau}_i^{(2)}\cdot\left(\sum_{j=1}^{N}\gamma_j^{(1)}\boldsymbol{\nu}_{ij}^{(1)}+\sum_{j=1}^{N}\gamma_j^{(2)}\boldsymbol{\nu}_{ij}^{(2)}\right)+\frac{\gamma_i^{(1)}}{2}=\frac{b_i^{(2)}}{A_i}.
\end{aligned} \tag{3}$$

Here

$$\boldsymbol{\nu}_{ij}^{(k)}=\int\limits_{K_i}\left(\int\limits_{K_j}\frac{\boldsymbol{\tau}_j^{(k)}\times(\boldsymbol{r}-\boldsymbol{\xi}\,)}{\left|\boldsymbol{r}-\boldsymbol{\xi}\,\right|^3}d\xi\right)dr,\quad b_i^{(k)}=\int\limits_{K_i}\boldsymbol{\tau}_i^{(k)}\cdot\boldsymbol{f}(\boldsymbol{r},\,t)\,dr,\quad k=1,\,2;\;i,j=1,\,\ldots,\,N.$$

Algebraic system (3) has infinite set of solutions; in order to select the unique solution we should satisfy additional condition

$$\int\limits_{K}\boldsymbol{\gamma}(\boldsymbol{r},\,t)dr=\boldsymbol{0},$$

which can be written down in the following form:

$$\sum_{i=1}^{N}A_i\left(\gamma_i^{(1)}\boldsymbol{\tau}_i^{(1)}+\gamma_i^{(2)}\boldsymbol{\tau}_i^{(2)}\right)=\boldsymbol{0}. \tag{4}$$

System (3)-(4) is overdetermined, it should be regularized, for example, by introducing the 3D-vector $\boldsymbol{R}=\left(R_1,\,R_2,\,R_3\right)^T$:

$$\begin{aligned}
&\frac{1}{4\pi A_i}\boldsymbol{\tau}_i^{(1)}\cdot\left(\sum_{j=1}^{N}\gamma_j^{(1)}\boldsymbol{\nu}_{ij}^{(1)}+\sum_{j=1}^{N}\gamma_j^{(2)}\boldsymbol{\nu}_{ij}^{(2)}\right)-\frac{\gamma_i^{(2)}}{2}+\boldsymbol{R}\cdot\boldsymbol{\tau}_i^{(2)}=\frac{b_i^{(1)}}{A_i},\\
&\frac{1}{4\pi A_i}\boldsymbol{\tau}_i^{(2)}\cdot\left(\sum_{j=1}^{N}\gamma_j^{(1)}\boldsymbol{\nu}_{ij}^{(1)}+\sum_{j=1}^{N}\gamma_j^{(2)}\boldsymbol{\nu}_{ij}^{(2)}\right)+\frac{\gamma_i^{(1)}}{2}+\boldsymbol{R}\cdot\boldsymbol{\tau}_i^{(1)}=\frac{b_i^{(2)}}{A_i},\\
&\sum_{j=1}^{N}A_j\left(\gamma_j^{(1)}\boldsymbol{\tau}_j^{(1)}+\gamma_j^{(2)}\boldsymbol{\tau}_j^{(2)}\right)=0,\qquad i=1,\,\ldots,\,N.
\end{aligned} \tag{5}$$

Numerical computations show that system (5) is well-conditioned; its dimension is $2N+3$.

4 MATRIX COEFFICIENTS CALCULATION

The main problem for practical use of the suggested approach is coefficients $\boldsymbol{\nu}_{ij}^{(k)}$ calculation for system (5):

$$\boldsymbol{\nu}_{ij}^{(k)} = \boldsymbol{\tau}_j^{(k)} \times \int\limits_{K_i} \left(\int\limits_{K_j} \frac{\boldsymbol{r} - \boldsymbol{\xi}}{\left| \boldsymbol{r} - \boldsymbol{\xi} \right|^3} d\xi \right) dr = \boldsymbol{\tau}_j^{(k)} \times \boldsymbol{I}_{ij}, \quad k = 1,\, 2, \quad i,\, j = 1,\, \ldots,\, N$$

Integral $\boldsymbol{I}_{ij}$ is calculated over triangular cells K_i and K_j, where i-th cell we call 'control', j-th cell — 'influence' cell.

The simplest way for integrals $\boldsymbol{I}_{ij}$ numerical computation is Gaussian quadrature formula usage. But this approach has high computational cost and leads to numerical calculation of improper integral when influence and control cells are neighbor (if they has common edge or common vertex).

The other way is to use some semi-analytical approaches. Firstly, the inner integral over the influence cell K_j can be calculated exactly:

$$\boldsymbol{J}_j(\boldsymbol{r}\,) = \int\limits_{K_j} \frac{\boldsymbol{r} - \boldsymbol{\xi}}{\left| \boldsymbol{r} - \boldsymbol{\xi} \right|^3} d\xi \tag{6}$$

Secondly, the outer integral over the control cell

$$\boldsymbol{I}_{ij} = \int\limits_{K_i} \boldsymbol{J}_j(\boldsymbol{r}\,)\, dr \tag{7}$$

can be calculated using Gaussian dots in non-neighbor case. If the cells have common edge, we need to exclude the singularity from the $\boldsymbol{J}_j(\boldsymbol{r}\,)$ and write down it in form

$$\boldsymbol{J}_j(\boldsymbol{r}\,) = \boldsymbol{J}_j^{\,\mathrm{reg}}(\boldsymbol{r}\,) + \boldsymbol{J}_j^{\,\mathrm{sing}}(\boldsymbol{r}\,).$$

where $\boldsymbol{J}_j^{\,\mathrm{reg}}(\boldsymbol{r}\,)$ has no singularities and can be easily integrated numerically with high accuracy, while the integral

$$\int\limits_{K_i} \boldsymbol{J}_j^{\,\mathrm{sing}}(\boldsymbol{r}\,)\, dr$$

can be calculated exactly in analytic form.

There is well-known way for analytical calculation of integral (6) via considering of the integral from $\left| \boldsymbol{r} - \boldsymbol{\xi} \right|^{-1}$ with respect to $\boldsymbol{\xi}$ over the triangle K_j:

$$\boldsymbol{J}_j(\boldsymbol{r}\,) = \int\limits_{K_j} \frac{\boldsymbol{r} - \boldsymbol{\xi}}{\left| \boldsymbol{r} - \boldsymbol{\xi} \right|^3} d\xi = \int\limits_{K_j} \nabla_\xi \frac{1}{\left| \boldsymbol{r} - \boldsymbol{\xi} \right|} d\xi = -\int\limits_{K_j} \nabla_r \frac{1}{\left| \boldsymbol{r} - \boldsymbol{\xi} \right|} d\xi = -\nabla_r \left(\int\limits_{K_j} \frac{1}{\left| \boldsymbol{r} - \boldsymbol{\xi} \right|} d\xi \right).$$

The last integral is very usual in potential theory, analytical expression for it can be found, for example, in [5]. However, that expression is cumbersome and it also should be differentiated with respect to components of vector $\boldsymbol{r}$.

Using computational software of symbolic mathematics *Wolfram Mathematica* and Handbook of integrals [6] it is possible to integrate (6) straightforwardly if vectors $\boldsymbol{s}_k = \boldsymbol{r}_k^{(j)} - \boldsymbol{r}$, $k = 1, 2, 3$, are only known, where $\boldsymbol{r}$ is point for which integral (6) is calculated, $\boldsymbol{r}_k^{(j)}$ are vertices of K_j triangular cell. Denoting

$$\boldsymbol{e}_k^{(j)} = \frac{\boldsymbol{s}_{k+1} - \boldsymbol{s}_k}{\left|\boldsymbol{s}_{k+1} - \boldsymbol{s}_k\right|} = \frac{\boldsymbol{r}_{k+1}^{(j)} - \boldsymbol{r}_k^{(j)}}{\left|\boldsymbol{r}_{k+1}^{(j)} - \boldsymbol{r}_k^{(j)}\right|}, \qquad \boldsymbol{\sigma}_k = \frac{\boldsymbol{s}_k}{\left|\boldsymbol{s}_k\right|}, \quad k = 1, 2, 3,$$

and assuming all the indices to be calculated using a modulus of 3, we obtain

$$\boldsymbol{J}_j(\boldsymbol{r}\,) = \Theta_j \boldsymbol{n}_j + \boldsymbol{\Psi}_j \times \boldsymbol{n}_j, \quad j = 1, \ldots, N,$$

where

$$\boldsymbol{\Psi}_j = \sum_{k=1}^{3} \ln\!\left(\frac{\left|\boldsymbol{s}_k\,\right|}{\left|\boldsymbol{s}_{k+1}\,\right|} \frac{1 + \boldsymbol{e}_k^{(j)} \cdot \boldsymbol{\sigma}_k}{1 + \boldsymbol{e}_k^{(j)} \cdot \boldsymbol{\sigma}_{k+1}} \right) \boldsymbol{e}_k,$$

Θ_j is solid angle subtended by triangular cell K_j which can be calculated, for example, by using the formula [7]

$$\Theta_j = 2 \arctan\!\left(\frac{\left|\boldsymbol{s}_1 \boldsymbol{s}_2 \boldsymbol{s}_3\,\right|}{\left|\boldsymbol{s}_1\,\right| \cdot \left|\boldsymbol{s}_2\,\right| \cdot \left|\boldsymbol{s}_3\,\right| + \left(\boldsymbol{s}_1 \cdot \boldsymbol{s}_2\,\right)\left|\boldsymbol{s}_3\,\right| + \left(\boldsymbol{s}_2 \cdot \boldsymbol{s}_3\,\right)\left|\boldsymbol{s}_1\,\right| + \left(\boldsymbol{s}_3 \cdot \boldsymbol{s}_1\,\right)\left|\boldsymbol{s}_2\,\right|} \right),$$

here $\boldsymbol{s}_1 \boldsymbol{s}_2 \boldsymbol{s}_3$ denotes the scalar triple product of the vectors.

The procedure of numerical calculation of the integral (7) depends on relative position of cells K_i and K_j. Three different cases should be considered: when the cells don't have common edges and vertices (non-neighbouring case), when cells have common edge (and two common vertices) and, finally, when they have common vertex only.

4.1 Non-neighbouring cells case

Calculation of the integral (7) when cells K_i and K_j don't have common edges and vertices, is provided numerically by using Gaussian quadrature formula:

$$\boldsymbol{I}_{ij} = \int_{K_i} \boldsymbol{J}_j(\boldsymbol{r}\,)\, dr \approx A_i \sum_{p=1}^{N_{GP}} \omega_p \boldsymbol{J}_j(\boldsymbol{\eta}_p\,),$$

where N_{GP} is number of Gaussian points; ω_p are weight coefficients; $\boldsymbol{\eta}_p$ are positions of Gaussian points. Values of ω_p and $\boldsymbol{\eta}_p$ for different values of N_{GP} can be found, for example, in [8].

4.2 Neighbouring cells with common edge case

If cells K_i and K_j have common edge with directing unit vector $\boldsymbol{e}_c$, for example as it is shown on fig. 1, we should apply Gaussian integration to regular part $\boldsymbol{J}_j^{\,\mathrm{reg}}(\boldsymbol{r}\,)$, which has the form

$$\boldsymbol{J}_j^{\,\mathrm{reg}}(\boldsymbol{r}\,) = \boldsymbol{J}_j(\boldsymbol{r}\,) - \left(\boldsymbol{e}_c \ln\Big(\frac{|\boldsymbol{a}\,|\cdot|\boldsymbol{c}\,| - \boldsymbol{a}\cdot\boldsymbol{c}}{|\boldsymbol{b}\,|\cdot|\boldsymbol{c}\,| - \boldsymbol{b}\cdot\boldsymbol{c}}\Big) + \boldsymbol{e}_1^{\,(j)} \ln\Big(\frac{\boldsymbol{a}\cdot\boldsymbol{c}}{|\boldsymbol{c}\,|^2}\Big) + \boldsymbol{e}_2^{\,(j)} \ln\Big(-\frac{\boldsymbol{b}\cdot\boldsymbol{c}}{|\boldsymbol{c}\,|^2}\Big)\right) \times \boldsymbol{n}_j,$$

or, the same written down in the other way,

$$\boldsymbol{J}_j^{\,\mathrm{reg}}(\boldsymbol{r}\,) = \boldsymbol{J}_j(\boldsymbol{r}\,) - \left(\boldsymbol{e}_c \ln\Big(\frac{|\boldsymbol{a}\,|(1-\cos\alpha)}{|\boldsymbol{b}\,|(1-\cos\beta)}\Big) + \boldsymbol{e}_1^{\,(j)} \ln\Big(\frac{|\boldsymbol{a}\,|}{|\boldsymbol{c}\,|}\cos\alpha\Big) + \boldsymbol{e}_2^{\,(j)} \ln\Big(\frac{|\boldsymbol{b}\,|}{|\boldsymbol{c}\,|}\cos\beta\Big)\right) \times \boldsymbol{n}_j.$$

Where

$$\boldsymbol{e}_c = \frac{\boldsymbol{c}}{|\boldsymbol{c}\,|} = \boldsymbol{e}_3^{\,(j)}, \quad \boldsymbol{c} = \boldsymbol{r}_1^{\,(j)} - \boldsymbol{r}_3^{\,(j)}, \quad \boldsymbol{a} = \boldsymbol{r}_1^{\,(j)} - \boldsymbol{r}, \quad \boldsymbol{b} = \boldsymbol{r}_3^{\,(j)} - \boldsymbol{r},$$

$$\cos\alpha = \frac{\boldsymbol{a}\cdot\boldsymbol{c}}{|\boldsymbol{a}\,|\cdot|\boldsymbol{c}\,|}, \quad \cos\beta = -\frac{\boldsymbol{b}\cdot\boldsymbol{c}}{|\boldsymbol{b}\,|\cdot|\boldsymbol{c}\,|}.$$

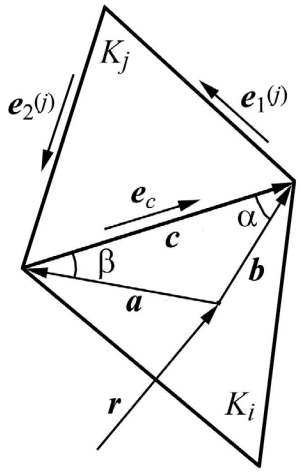

Figure 1: Cells K_i and K_j in case of having common edge

Denoting

$$\boldsymbol{\Psi}_j^{\,\mathrm{sing}} = \boldsymbol{e}_c \ln\left(\frac{|\boldsymbol{a}\,|(1-\cos\alpha)}{|\boldsymbol{b}\,|(1-\cos\beta)}\right) + \boldsymbol{e}_1^{\,(j)} \ln\left(\frac{|\boldsymbol{a}\,|}{|\boldsymbol{c}\,|}\cos\alpha\right) + \boldsymbol{e}_2^{\,(j)} \ln\left(\frac{|\boldsymbol{b}\,|}{|\boldsymbol{c}\,|}\cos\beta\right),$$

we finally obtain

$$\boldsymbol{J}_j^{\,\mathrm{reg}}(\boldsymbol{r}\,) = \Theta_j \boldsymbol{n}_j + (\boldsymbol{\Psi}_j - \boldsymbol{\Psi}_j^{\,\mathrm{sing}}) \times \boldsymbol{n}_j, \quad \boldsymbol{J}_j^{\,\mathrm{sing}}(\boldsymbol{r}\,) = \boldsymbol{\Psi}_j^{\,\mathrm{sing}} \times \boldsymbol{n}_j.$$

All scalar multipliers of $\boldsymbol{\Psi}_j^{\text{sing}}$ can be integrated analytically over the cell K_i:

$$q_c = \int\limits_{K_i} \ln\left(\frac{|\boldsymbol{a}\,|(1-\cos\alpha)}{|\boldsymbol{b}\,|(1-\cos\beta)}\right)dr =$$
$$= \frac{|\boldsymbol{c}\,|^2}{2}\left(\sin\alpha\,\ln\frac{|\boldsymbol{c}\,|^2\cos^2(\frac{\alpha_i}{2})\sin\beta_i}{2A_i\cos^2(\frac{\alpha_i+\beta_i}{2})} + \sin\beta\,\ln\frac{|\boldsymbol{c}\,|^2\cos^2(\frac{\beta_i}{2})\sin\alpha_i}{2A_i\cos^2(\frac{\alpha_i+\beta_i}{2})}\right) - A_i\ln\left(\tan\frac{\alpha_i}{2}\tan\frac{\beta_i}{2}\right),$$
$$q_1 = \int\limits_{K_i} \ln\left(\frac{|\boldsymbol{a}\,|}{|\boldsymbol{c}\,|}\cos\alpha\right)dr = -\frac{A_i}{2}\left(3 - 2\frac{\tan\beta_i}{\tan\alpha_i}\ln\left(1+\frac{\tan\alpha_i}{\tan\beta_i}\right)\right),$$
$$q_2 = \int\limits_{K_i} \ln\left(\frac{|\boldsymbol{b}\,|}{|\boldsymbol{c}\,|}\cos\beta\right)dr = -\frac{A_i}{2}\left(3 - 2\frac{\tan\alpha_i}{\tan\beta_i}\ln\left(1+\frac{\tan\beta_i}{\tan\alpha_i}\right)\right).$$

Here α_i and β_i are angles of the triangle K_i, which adjoin the common edge of the cells K_i and K_j. Finally,

$$\boldsymbol{I}_{ij} = \int\limits_{K_i} \boldsymbol{J}_j^{\text{reg}}(\boldsymbol{r}\,)\,dr + \int\limits_{K_i} \boldsymbol{J}_j^{\text{sing}}(\boldsymbol{r}\,)\,dr \approx \left(A_i\sum_{p=1}^{N_{GP}}\omega_p\boldsymbol{J}_j^{\text{reg}}(\boldsymbol{\eta}_p)\right) + \left(\boldsymbol{e}_c q_c + \boldsymbol{e}_1^{(j)} q_1 + \boldsymbol{e}_2^{(j)} q_2\right)\times\boldsymbol{n}_j.$$

4.3 Neighbouring cells with common vertex case

If cells K_i and K_j have common vertex, for example as it is shown on fig. 2, the regular part $\boldsymbol{J}_j^{\text{reg}}(\boldsymbol{r}\,)$ has the following form (the previous denotation is used):

$$\boldsymbol{J}_j^{\text{reg}}(\boldsymbol{r}) = \boldsymbol{J}_j(\boldsymbol{r}) - \ln\left(\frac{|\boldsymbol{a}|}{\sqrt{A_j}}\right)\left(\boldsymbol{e}_1^{(j)} + \boldsymbol{e}_3^{(j)}\right)\times\boldsymbol{n}_j$$

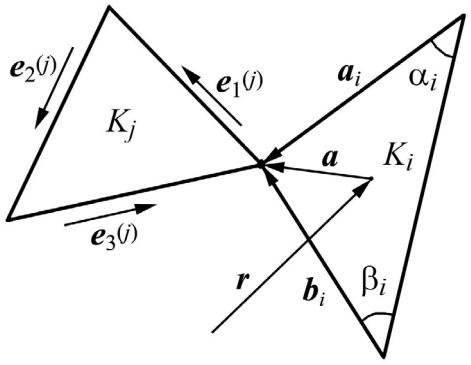

Figure 2: Cells K_i and K_j in case of having common vertex

Again, denoting

$$\boldsymbol{\Psi}_j^{\text{sing}} = \left(\boldsymbol{e}_1^{(j)} + \boldsymbol{e}_3^{(j)}\right)\ln\left(\frac{|\boldsymbol{a}\,|}{\sqrt{A_j}}\right),$$

we obtain

$$\boldsymbol{J}_j^{\,\mathrm{reg}}(\boldsymbol{r}\,) = \Theta_j \boldsymbol{n}_j + (\boldsymbol{\Psi}_j - \boldsymbol{\Psi}_j^{\,\mathrm{sing}}) \times \boldsymbol{n}_j, \quad \boldsymbol{J}_j^{\,\mathrm{sing}}(\boldsymbol{r}\,) = \boldsymbol{\Psi}_j^{\,\mathrm{sing}} \times \boldsymbol{n}_j.$$

The scalar multiplier in $\boldsymbol{\Psi}_j^{\,\mathrm{sing}}$ can be integrated analytically over the cell K_i:

$$q = \int\limits_{K_i} \ln\!\left(\frac{|\boldsymbol{a}\,|}{\sqrt{A_j}}\right) dr = \\ = \frac{1}{4}\left(2A_i\left(\ln\frac{A_i}{A_j} - 3\right) + 2(\pi - \alpha_i - \beta_i)|\boldsymbol{a}_i| \cdot |\boldsymbol{b}_i| \sin\alpha_i \sin\beta_i + \right. \\ \left. + |\boldsymbol{a}_i|^2 \sin 2\alpha_i \ln\frac{|\boldsymbol{a}_i|^2}{A_i} + |\boldsymbol{b}_i|^2 \sin 2\beta_i \ln\frac{|\boldsymbol{b}_i|^2}{A_i}\right).$$

Here α_i and β_i are angles of the triangle K_i, which don't adjoin the common vertex of the cells K_i and K_j. Finally,

$$\boldsymbol{I}_{ij} = \int\limits_{K_i} \boldsymbol{J}_j^{\,\mathrm{reg}}(\boldsymbol{r}\,)\, dr + \int\limits_{K_i} \boldsymbol{J}_j^{\,\mathrm{sing}}(\boldsymbol{r}\,)\, dr \approx \left(A_i \sum_{p=1}^{N_{GP}} \omega_p \boldsymbol{J}_j^{\,\mathrm{reg}}(\boldsymbol{\eta}_p)\right) + q\left(\boldsymbol{e}_1^{\,(j)} + \boldsymbol{e}_3^{\,(j)}\right) \times \boldsymbol{n}_j.$$

5 CONCLUSIONS

The dirived formulae for $\boldsymbol{I}_{ij}$ make it possible to construct numerical procedure for solving of the discrete analogue of the integral equation for vortex sheet intensity calculation in the framework of 'tangent' approach. The analogue formulae can be easily obtained for vortex element upon boundary cells influence calculation as well as cells upon vortex element calculation. It allows to use arbitrary triangular mesh on body surface and to refine mesh near sharp edges, that is especially important for flow around 3D wings simulation.

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